Subscription Dynamics and Competition in Communications Markets

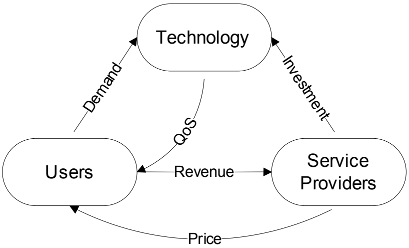
Shaolei Ren, Jaeok Park, and Mihaela van der Schaar

Electrical Engineering Department
University of California, Los Angeles

October 3, 2010

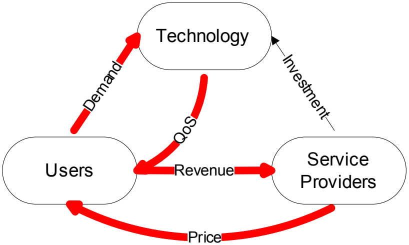
Outline

Overview of Communications Markets



Interaction among technology, users and service providers

Our Work

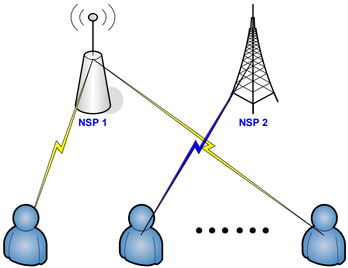


How does the technology influence the users' demand and the service providers' revenues?

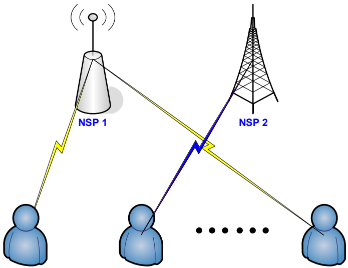
- We consider a duopoly communications market.
- Given prices, how does QoS affect the subscription decisions (or demand) of users?
- How are prices determined through competition between the service providers?

Outline

Model



Model



Network model

- network service providers: $\mathcal{S}_1$ and $\mathcal{S}_2$
- continuum model: a large number of users

Model

Service providers

- $\mathcal{S}_i$: price p_i and fraction of subscribers $\lambda_i(p_i, p_{-i})$

Model

Service providers

- $\mathcal{S}_i$: price p_i and fraction of subscribers $\lambda_i(p_i, p_{-i})$
- utility (revenue): $\mathcal{R}_i(p_i, p_{-i}) = p_i \lambda_i(p_i, p_{-i})$

Model

Service providers

- $\mathcal{S}_i$: price p_i and fraction of subscribers $\lambda_i(p_i, p_{-i})$
- utility (revenue): $\mathcal{R}_i(p_i, p_{-i}) = p_i \lambda_i(p_i, p_{-i})$

Users

- user k: $u_k = \alpha_k q_i - p_i$ if it subscribes to $\mathcal{S}_i$

Model

Service providers

- $\mathcal{S}_i$: price p_i and fraction of subscribers $\lambda_i(p_i, p_{-i})$
- utility (revenue): $\mathcal{R}_i(p_i, p_{-i}) = p_i \lambda_i(p_i, p_{-i})$

Users

- user k: $u_k = \alpha_k q_i - p_i$ if it subscribes to $\mathcal{S}_i$
- α_k follows a distribution with PDF $f(\alpha)$

Model

Service providers

- $\mathcal{S}_i$: price p_i and fraction of subscribers $\lambda_i(p_i, p_{-i})$
- utility (revenue): $\mathcal{R}_i(p_i, p_{-i}) = p_i \lambda_i(p_i, p_{-i})$

Users

- user k: $u_k = \alpha_k q_i - p_i$ if it subscribes to $\mathcal{S}_i$
- α_k follows a distribution with PDF $f(\alpha)$

assumptions on $f(\alpha)$

- $f(\alpha) > 0$ if $\alpha \in [0, \beta]$ and $f(\alpha) = 0$ otherwise
- $f(\alpha)$ is continuous on $[0, \beta]$

Model

Service providers

- $\mathcal{S}_i$: price p_i and fraction of subscribers $\lambda_i(p_i, p_{-i})$
- utility (revenue): $\mathcal{R}_i(p_i, p_{-i}) = p_i \lambda_i(p_i, p_{-i})$

Users

- user k: $u_k = \alpha_k q_i - p_i$ if it subscribes to $\mathcal{S}_i$
- α_k follows a distribution with PDF $f(\alpha)$

QoS model

- q_1 is constant
- $q_2 = g(\lambda_2)$, where $g(\lambda_2) \in (0, q_1)$ is a differentiable and non-increasing function of $\lambda_2 \in [0, 1]$

Outline

User Subscription

- Discrete-time model $\{(\lambda_1^t, \lambda_2^t) \mid t = 0, 1, 2 \cdots\}$
- Users' belief model and subscription decisions
  - naive (or static) expectation: every user expects that the QoS in the current period is equal to that in the previous period (i.e., $\tilde{g}_k(\lambda_2^t) = g(\lambda_2^{t-1})$)
  - a user subscribes to whichever NSP provides a higher (non-negative) utility

User Subscription

- Discrete-time model $\{(\lambda_1^t, \lambda_2^t) \mid t = 0, 1, 2 \cdots\}$
- Users' belief model and subscription decisions
  - naive (or static) expectation: every user expects that the QoS in the current period is equal to that in the previous period (i.e., $\tilde{g}_k(\lambda_2^t) = g(\lambda_2^{t-1})$)
  - a user subscribes to whichever NSP provides a higher (non-negative) utility
- Dynamics of user subscriptions

User Subscription

- Discrete-time model $\{(\lambda_1^t, \lambda_2^t) \mid t = 0, 1, 2 \cdots\}$
- Users' belief model and subscription decisions
  - naive (or static) expectation: every user expects that the QoS in the current period is equal to that in the previous period (i.e., $\tilde{g}_k(\lambda_2^t) = g(\lambda_2^{t-1})$)
  - a user subscribes to whichever NSP provides a higher (non-negative) utility
- Dynamics of user subscriptions

if $\frac{p_1}{q_1} > \frac{p_2}{g(\lambda_2^{t-1})}$, then

$$\begin{aligned} \lambda_1^t = \quad & h_{d,1}(\lambda_1^{t-1}, \lambda_2^{t-1}) = \quad 1 - F\left(\frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})}\right), \\ \lambda_2^t = \quad & h_{d,2}(\lambda_1^{t-1}, \lambda_2^{t-1}) = \quad F\left(\frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})}\right) - F\left(\frac{p_2}{g(\lambda_2^{t-1})}\right) \end{aligned}$$

User Subscription

- Discrete-time model $\{(\lambda_1^t, \lambda_2^t) \mid t = 0, 1, 2 \cdots\}$
- Users' belief model and subscription decisions
  - naive (or static) expectation: every user expects that the QoS in the current period is equal to that in the previous period (i.e., $\tilde{g}_k(\lambda_2^t) = g(\lambda_2^{t-1})$)
  - a user subscribes to whichever NSP provides a higher (non-negative) utility
- Dynamics of user subscriptions

if $\frac{p_1}{q_1} \leq \frac{p_2}{g(\lambda_2^{t-1})}$, then

$$\begin{aligned} \lambda_1^t = & \quad h_{d,1}(\lambda_1^{t-1}, \lambda_2^{t-1}) = & 1 - F\left(\frac{p_1}{q_1}\right), \\ \lambda_2^t = & \quad h_{d,2}(\lambda_1^{t-1}, \lambda_2^{t-1}) = & 0. \end{aligned}$$

Equilibrium Analysis

- Stabilized fraction of subscribers will stabilize in the long run

Definition

$(\lambda_1^*, \lambda_2^*)$ is an *equilibrium* point of the user subscription dynamics in the duopoly market if it satisfies $h_{d,1}(\lambda_1^*, \lambda_2^*) = \lambda_1^*$ and $h_{d,2}(\lambda_1^*, \lambda_2^*) = \lambda_2^*$.

Equilibrium Analysis

- Stabilized fraction of subscribers will stabilize in the long run

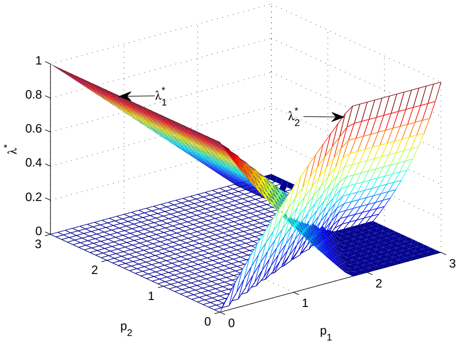
Proposition (uniqueness and existence of $(\lambda_1^*, \lambda_2^*)$)

For any non-negative price pair (p_1, p_2), there exists a unique equilibrium point $(\lambda_1^*, \lambda_2^*)$ of the user subscription dynamics in the duopoly market. Moreover, $(\lambda_1^*, \lambda_2^*)$ satisfies

$$\begin{cases} \lambda_1^* = 1 - F\left(\frac{p_1}{q_1}\right), \lambda_2^* = 0, & \text{if } \frac{p_1}{q_1} \leq \frac{p_2}{g(0)}, \\ \lambda_1^* = 1 - F\left(\theta_1^*\right), \lambda_2^* = F\left(\theta_1^*\right) - F\left(\theta_2^*\right), & \text{if } \frac{p_1}{q_1} > \frac{p_2}{g(0)}, \end{cases}$$

where $\theta_1^* = (p_1 - p_2)/(q_1 - g(\lambda_2^*))$ and $\theta_2^* = p_2/g(\lambda_2^*)$.

Equilibrium Market Shares



- $q_1 = 2.5$, $g(\lambda_2) = 1.2e^{-0.5\lambda_2}$, and α is uniformly distributed on $[0, 1]$, i.e., $f_a(\alpha) = 1$ for $\alpha \in [0, 1]$.

Convergence of User Subscription Dynamics

- Convergence is not always guaranteed

Convergence of User Subscription Dynamics

- Convergence is not always guaranteed

Example: when the QoS of NSP $\mathcal{S}_2$ degrades fast w.r.t. the fraction of subscribers

1. suppose that only a small fraction of users subscribe to NSP $\mathcal{S}_2$ at period t and each subscriber obtains a high QoS
2. a large fraction of users subscribe at period $t+1$, which will result in a low QoS at period $t+1$
3. a small fraction of subscribers at period $t+2$

Convergence of User Subscription Dynamics

- Convergence is not always guaranteed

Theorem

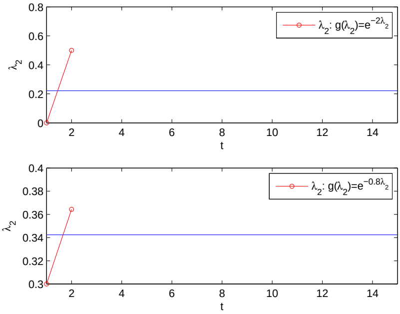
For any non-negative price pair (p_1, p_2)*, the user subscription dynamics converges to the unique equilibrium point starting from any initial point* $(\lambda_1^0, \lambda_2^0) \in \Lambda$ *if*

$$\max_{\lambda_2 \in [0,1]} \left\{ -\frac{g'(\lambda_2)}{g(\lambda_2)} \cdot \frac{q_1}{q_1 - g(\lambda_2)} \right\} < \frac{1}{K},$$

where $K = \max_{\alpha \in [0,\beta]} f(\alpha)\alpha$.

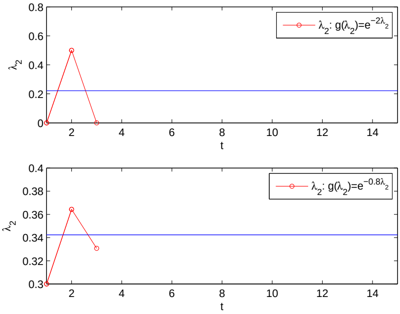
proof

Illustration of Oscillation & Convergence



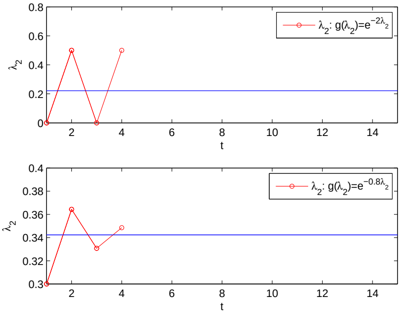
$$t = 2$$

Illustration of Oscillation & Convergence



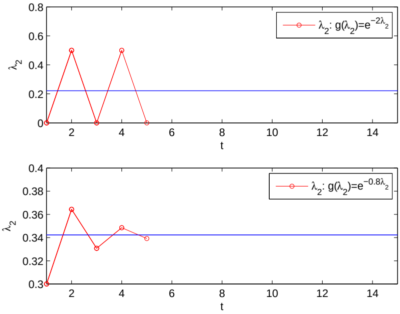
$$t = 3$$

Illustration of Oscillation & Convergence



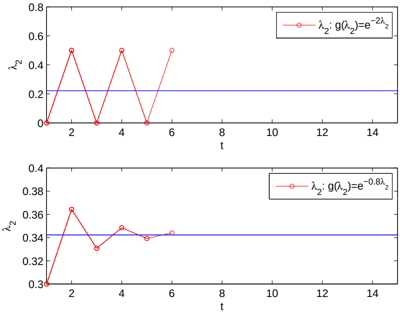
$t = 4$

Illustration of Oscillation & Convergence



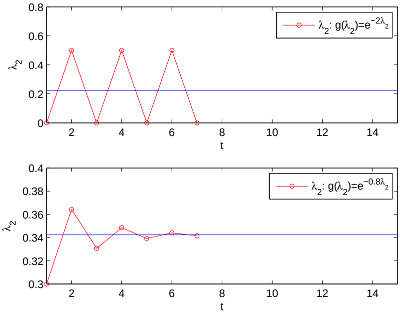
$$t = 5$$

Illustration of Oscillation & Convergence



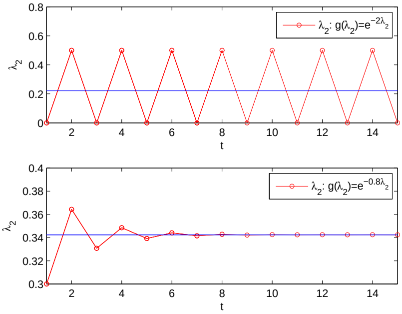
$t = 6$

Illustration of Oscillation & Convergence



$t = 7$

Illustration of Oscillation & Convergence



$t = 15$

Outline

Cournot Competition

- We model competition between the NSPs using Cournot competition.
  - each NSP chooses the fraction of subscribers independently
  - prices are determined such that the equilibrium market shares equate the chosen quantities

Cournot Competition

- We model competition between the NSPs using Cournot competition.
  - each NSP chooses the fraction of subscribers independently
  - prices are determined such that the equilibrium market shares equate the chosen quantities
- $\mathcal{G}_C = \{\mathcal{S}_i, R_i(\lambda_1, \lambda_2), \lambda_i \in [0,1) \mid i = 1, 2\}$

Cournot Competition

- We model competition between the NSPs using Cournot competition.
  - each NSP chooses the fraction of subscribers independently
  - prices are determined such that the equilibrium market shares equate the chosen quantities
- $\mathcal{G}_C = \{\mathcal{S}_i, R_i(\lambda_1, \lambda_2), \lambda_i \in [0, 1) \mid i = 1, 2\}$
- $(\lambda_1^{**}, \lambda_2^{**})$ is a (pure) NE of $\mathcal{G}_C$ (or a Cournot equilibrium) if it satisfies

$$R_i(\lambda_i^{**}, \lambda_{-i}^{**}) \geq R_i(\lambda_i, \lambda_{-i}^{**}),\ \forall\, \lambda_i \in [0, 1), \forall\, i = 1, 2\,.$$

Existence of NE

Lemma

Suppose that $f(\cdot)$ is non-increasing on $[0, \beta]$. Let $\tilde{\lambda}_i(\lambda_{-i})$ be a market share that maximizes the revenue of NSP $\mathcal{S}_i$ provided that NSP $\mathcal{S}_{-i}$ chooses $\lambda_{-i} \in [0, 1)$, i.e., $\tilde{\lambda}_i(\lambda_{-i}) \in \arg\max_{\lambda_i \in [0,1)} R_i(\lambda_i, \lambda_{-i})$. Then $\tilde{\lambda}_i(\lambda_{-i}) \in (0, 1/2]$ for all $\lambda_{-i} \in [0, 1)$, for all $i = 1, 2$. Moreover, $\tilde{\lambda}_i(\lambda_{-i}) \neq 1/2$ if $\lambda_{-i} > 0$, for $i = 1, 2$.

Existence of NE

Lemma

Suppose that $f(\cdot)$ is non-increasing on $[0, \beta]$. Let $\tilde{\lambda}_i(\lambda_{-i})$ be a market share that maximizes the revenue of NSP $\mathcal{S}_i$ provided that NSP $\mathcal{S}_{-i}$ chooses $\lambda_{-i} \in [0, 1)$, i.e., $\tilde{\lambda}_i(\lambda_{-i}) \in \arg\max_{\lambda_i \in [0,1)} R_i(\lambda_i, \lambda_{-i})$. Then $\tilde{\lambda}_i(\lambda_{-i}) \in (0, 1/2]$ for all $\lambda_{-i} \in [0, 1)$, for all $i = 1, 2$. Moreover, $\tilde{\lambda}_i(\lambda_{-i}) \neq 1/2$ if $\lambda_{-i} > 0$, for $i = 1, 2$.

- Implication
  - when the strategy space is specified as $[0, 1)$ and $f(\cdot)$ satisfies the non-increasing property, strategies $\lambda_i \in \{0\} \cup (1/2, 1)$ is strictly dominated for $i = 1, 2$
  - if a NE $(\lambda_1^{**}, \lambda_2^{**})$ of $\tilde{\mathcal{G}}_C$ exists, then it must satisfy $(\lambda_1^{**}, \lambda_2^{**}) \in (0, 1/2)^2$

Existence of NE

Theorem

Suppose that $f(\cdot)$ is non-increasing and continuously differentiable on $[0, \beta]$. If $f(\cdot)$ and $g(\cdot)$ satisfy some conditions (Eqn. 18 and Eqn. 19 in the paper), then the game $\tilde{\mathcal{G}}_C$ has at least one NE.

Existence of NE

Corollary

Suppose that the users' valuation of QoS is uniformly distributed, i.e., $f(\alpha) = 1/\beta$ *for* $\alpha \in [0, \beta]$*. If* $g(\lambda_2) + \lambda_2 g'(\lambda_2) \geq 0$ *for all* $\lambda_2 \in [0, 1/2]$*, then the game* $\mathcal{G}_C$ *has at least one NE.*

- Interpretation
  - if the elasticity of the QoS provided by NSP $\mathcal{S}_2$ with respect to the fraction of its subscribers is no larger than 1 (i.e., $-[g'(\lambda_2)\lambda_2/g(\lambda_2)] \leq 1$), the Cournot competition game with the strategy space $[0, 1)$ has at least one NE
  - the condition is analogous to the sufficient conditions for convergence in that it requires that the QoS provided by NSP $\mathcal{S}_2$ cannot degrade too fast with respect to the fraction of subscribers.

Outline

Numerical Results

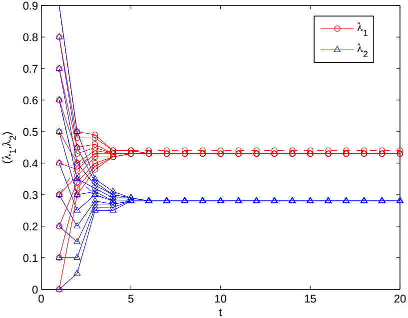


Figure: Dynamics of market shares under the best-response dynamics. Solid: $g(\lambda_2) = 1 - \frac{\lambda_2}{8}$; dashed: $g(\lambda_2) = 1 - \frac{\lambda_2}{2}$.

Numerical Results

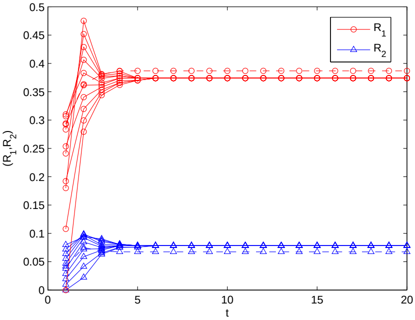


Figure: Iteration of revenues under the best-response dynamics. Solid: $g(\lambda_2) = 1 - \frac{\lambda_2}{8}$; dashed: $g(\lambda_2) = 1 - \frac{\lambda_2}{2}$.

Outline

Conclusion

Study the impacts of technologies on the user subscription dynamics

- constructed the dynamics of user subscription based on myopic updates
- showed that the existence of a unique equilibrium point of the user subscription dynamics
- provided a sufficient condition for the convergence of the user subscription dynamics: the QoS provided by NSP $\mathcal{S}_2$ should not degrade too fast as more users subscribe

Conclusion

Study the impacts of technologies on the user subscription dynamics

- constructed the dynamics of user subscription based on myopic updates
- showed that the existence of a unique equilibrium point of the user subscription dynamics
- provided a sufficient condition for the convergence of the user subscription dynamics: the QoS provided by NSP $\mathcal{S}_2$ should not degrade too fast as more users subscribe

Study the impacts of technologies on competition between the NSPs

- modeled the NSPs as strategic players in a non-cooperative Cournot game
- provided a sufficient condition that ensures the existence of at least one NE of the game

Selected References

- J. Musacchio and D. Kim, “Network platform competition in a two-sided market: implications to the net neutrality issue,” TPRC: Conference on Communication, Information, and Internet Policy, Sep. 2009.
- D. Bertsimas and G. Perakis, “Dynamic pricing: a learning approach,” Models for Congestion Charging/Network Pricing, 2005.
- G. Bitran and R. Galdentey, “An overview of pricing models for revenue management,” Manufacturing & Service Operational Management, vol. 5, no. 3, pp. 203-229, Summer 2003.
- F. Kelly, “Charging and rate control for elastic traffic,” European Transactions on Telecommunications, vol. 8, pp. 33-37, 1997.

Related Publications

- S. Ren, J. Park, and M. van der Schaar, "Dynamics of Service Provider Selection in Communication Markets," accepted and to appear in *Proc. IEEE Globecom* 2010.
- S. Ren, J. Park, and M. van der Schaar, "Subscription Dynamics and Competition in Communication Markets," in *Proc. ACM NetEcon* 2010.
- S. Ren, J. Park, and M. van der Schaar, "User Subscription Dynamics and Revenue Maximization in Communication Markets," *UCLA Tech. Report*, Aug. 2010 (available at http://arxiv.org/abs/1008.5367).

Convergence of User Subscription Dynamics

Proof.

1. Show that

$$\begin{aligned}
&\left\|\mathbf{h}_d(\lambda_{1,a},\lambda_{2,a})-\mathbf{h}_d(\lambda_{1,b},\lambda_{2,b})\right\|_\infty \\
&= K\left[-\frac{g'(\lambda_{2,c})}{g(\lambda_{2,c})}\cdot\frac{q_1}{q_1-g(\lambda_{2,c})}\right]\left|\lambda_{2,a}-\lambda_{2,b}\right| \\
&\le \kappa_d\left\|\lambda_a-\lambda_b\right\|_\infty .
\end{aligned}$$

where $\kappa_d = K \cdot \max_{\lambda_2\in[0,1]}\left\{\left[-g'(\lambda_2)/g(\lambda_2)\right]\cdot\left[q_1/(q_1-g(\lambda_2))\right]\right\}$

□

Convergence of User Subscription Dynamics

Proof.

1. Show that

$$\begin{aligned}
&\left\|\mathbf{h}_d(\lambda_{1,a},\lambda_{2,a})-\mathbf{h}_d(\lambda_{1,b},\lambda_{2,b})\right\|_\infty \\
&= K\left[-\frac{g'(\lambda_{2,c})}{g(\lambda_{2,c})}\cdot\frac{q_1}{q_1-g(\lambda_{2,c})}\right]\left|\lambda_{2,a}-\lambda_{2,b}\right| \\
&\leq \kappa_d\left\|\lambda_a-\lambda_b\right\|_\infty .
\end{aligned}$$

   where $\kappa_d = K\cdot\max_{\lambda_2\in[0,1]}\{[-g'(\lambda_2)/g(\lambda_2)]\cdot[q_1/(q_1-g(\lambda_2))]\}$

2. If $\max_{\lambda_2\in[0,1]}\left\{-\frac{g'(\lambda_2)}{g(\lambda_2)}\cdot\frac{q_1}{q_1-g(\lambda_2)}\right\}<\frac{1}{K}$, then the mapping is contraction!

back

□