# Matrix Methods for Lost Data Reconstruction in Erasure Codes

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## Abstract

Erasures codes, particularly those protecting against multiple failures in RAID disk arrays, provide a code-specific means for reconstruction of lost (erased) data. In the RAID application this is modeled as loss of strips so that reconstruction algorithms are usually optimized to reconstruct entire strips; that is, they apply only to highly correlated sector failures, i.e., sequential sectors on a lost disk. In this paper we address two more general problems: (1) recovery of lost data due to scattered or uncorrelated erasures and (2) recovery of partial (but sequential) data from a single lost disk (in the presence of any number of failures). The latter case may arise in the context of host IO to a partial strip on a lost disk. The methodology we propose for both problems is completely general and can be applied to any erasure code, but is most suitable for XOR-based codes.

For the scattered erasures, typically due to hard errors on the disk (or combinations of hard errors and disk loss), our methodology provides for one of two outcomes for the data on each lost sector. Either the lost data is declared unrecoverable (in the information-theoretic sense) or it is declared recoverable *and* a formula is provided for the reconstruction that depends only on readable sectors. In short, the methodology is both complete and constructive.

## 1 Introduction

XOR-based erasures codes for disk arrays model lost data most coarsely as loss of entire disks but more precisely as loss of entire symbols of the code. In practice, a symbol typically maps to a "strip", that is, multiple sequential sectors with one bit of the symbol corresponding to one or (typically) more sectors and with each different symbol residing on a different disk (this is not always the case, but it is a common practice). The collection of related strips is called a "stripe". To deal with disk failures, each erasure code comes complete with a specific reconstruction algorithm that is highly dependent on the code construction. For example, the 2-fault-tolerant X-code [10] is constructed geometrically, with parity values computed along diagonal paths through the data sectors. When two disks fail, the reconstruction follows these diagonal paths, starting at some initial point; that is, the reconstruction is both geometrically and recursively defined. The BCP [1] code is less geometrically designed, but still has a recursive reconstruction algorithm. More examples are mentioned in Section 2.

Erasures then are seen as correlated sector failures: all sectors in a strip are "lost" when the disk fails. However, increasing disk capacity together with a fairly stable bit-error rate implies that there is a significant probability of multiple uncorrelated or scattered *sector* errors within a given stripe, particularly in conjunction with one or more disk failures. For example, two disk losses plus a sector loss may occur often enough that even a two-disk fault tolerant code may not provide sufficient reliability. If all correlated and uncorrelated erasures occur within at most $t$ disks where $t$ is the (disk) fault tolerance of the code, then one method is to simulate loss of all affected disks and rebuild according to the code-specific reconstruction algorithm. However, this has two drawbacks. First, it is clear that this can be highly inefficient since it requires either reconstruction of "known" or readable data or it requires checking at each step of the process to see if a reconstruction is required. More importantly, however, this approach does not solve the more general problem when *more* than $t$ disks have been affected with sector losses. In such a case, it is quite possible that some or all of the lost sectors can be reconstructed, though this is not obvious *a priori* from the erasure correcting power of the code. For example, the 2-fault tolerant EVENODD code only claims to recover from two lost disks, so any additional sector loss typically means all lost data is declared unrecoverable. In fact, on average, anywhere from 40-60% of the lost sectors may be recovered in this situation.

In addition, while each erasure code provides a means to reconstruct entire strips (e.g., during a rebuild operation), to our knowledge, the literature does not contain any methods that explicitly address the problem of reconstructing a partial strip of lost data; such a need may arise in a host read operation to a failed disk during an incomplete rebuild operation. Of course, the strip reconstruction method could be applied in this case, but it is likely that such reconstruction will recover additional unnecessary lost sectors; that is, do more work than is

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required to service the host read, thereby adversely affecting performance. (This extra work may be worth the performance penalty in that the additional recovered sectors can be cached or added to the rebuild log, but that may not always be a desirable option.)

In this paper, we address both these problems. Our methodology is universal in that it can be applied to any erasure code of any fault tolerance. It applies to any failure scenario from full disk to scattered sectors to combinations of the two. It is based solely on the generator matrix for the erasure code. Consequently, a general erasure code reconstruction module could implement this methodology and use the generator matrix as one of its inputs. To emphasize the point, we address the problem of arbitrary sector (bit) erasures for *any* code designed with a strip (symbol) erasure failure model.

For the first problem of scattered (correlated and/or uncorrelated) sector loss, our methodology provides a mathematical guarantee: for each lost sector, either that sector's data is declared as (information-theoretically) unrecoverable (that is, a "data loss event") *or* the sector's data is declared recoverable and a reconstruction formula is generated. The reconstruction formula is a linear equation (XOR equation in case of XOR-based codes) involving known or readable data and parity sectors. In this respect, our methodology is both complete, constructive and universally applicable. It provides the best guarantee to meet the following requirement:

**User Contract**: *For any erasure scenario, the storage system shall recover any and all lost data sectors that the erasure code is information-theoretically capable of recovering.*

It should be noted that for 1-fault tolerant codes (e.g., RAID1, RAID4 or RAID5), the solution to both these problems is quite simple and obvious. Similarly, for Reed-Solomon codes [9] where the symbol is mapped to bytes or words (not sets of sectors), the standard Reed-Solomon procedure addresses both problems directly as well. The more interesting cases then are non-Reed-Solomon multiple fault-tolerant codes. Such codes are typically XOR-based as those have the most practical application. Careful and complex analysis of a specific code may produce a solution to this problem (and to our second problem). However, our solutions are universal. It is also clear that our methods can be extended to more general codes (e.g., some of the non-XOR codes in [3]). Furthermore, this methodology can be applied not just for RAID controllers but any application of these types of erasure codes such as dRAID (distributed Redundant Arrangement of Independent Devices) node-based systems.

For the second problem of partial strip reconstruction, we propose a hybrid solution: combine the inherent recursive method of the erasure code for full rebuild with the methodology for recovering scattered sectors. We also propose an alternative that is in many cases equivalent to the code-specific method, better in some cases and universally applicable to any erasure code.

Our methodology is based on principles of matrix theory and pseudo-inverses. Many codes (see [8, 9]) use full inverses to prove both that their codes have the declared fault tolerance and to provide reconstruction formulas. However, they apply it only to recover full code symbols, under maximal failures (where unique inverses exist) and not to the more general bit-within-a-symbol (*a.k.a*, sector within a strip) level that we address in this work.

The paper is organized as follows. We close the introduction with some definitions. The next section contains a few remarks on related work. Section 3 contains a brief review of the concepts from linear algebra that we need, particularly the notion of pseudo-inverse. In Section 4 we present a brief description of the generator matrix and parity check matrix for an erasure code. Section 5 explains how we simulate scattered sector loss and how we determine reconstructability in addressing our first problem. Section 6 contains algorithms for constructing pseudo-inverse matrices. We develop our methods in a detailed example in Section 7. Section 8 outlines the hybrid method for partial strip reconstruction (our second problem) and includes experimental results. We conclude with a brief summary.

## 1.1 Vocabulary

**sector:** the smallest unit of IO to/from a disk (typically 512 bytes at the disk drive, but perhaps 4KB from the filesystem or application layer).

**element:** a fundamental unit of data or parity; this is the building block of the erasure code. In coding theory, this is the data that is assigned to a bit within a symbol. We assume for simplicity that each element corresponds to a single sector; the more general case can be derived from this case.

**stripe:** a complete (connected) set of data and parity elements that are dependently related by parity computation relations. In coding theory, this is a code word; we use "code instance" synonymously.

**strip:** a unit of storage consisting of all contiguous elements (data, parity or both) from the same disk and stripe. In coding theory, this is associated with a code symbol. It is sometimes called a stripe unit. The set of strips in a code instance form a stripe. Typically, the strips are all of the same size (contain the same number of elements).

**array** A collection of disks on which one or more instances of a RAID erasure code is implemented.

## 2 Related Work

The two main results of this paper are (a) the application of pseudo-inverses of matrices to the problem of reconstruction of uncorrelated lost sectors and (b) a hybrid reconstruction that combines code-specific recursive reconstruction methods with this matrix method to efficiently reconstruct partial strips. To our knowledge neither of these problems has been specifically addressed in the literature. As remarked before, the theory of matrix inverses is used in the proof that some codes meet their declared strip (i.e., symbol) fault tolerance. For example, the Reed-Solomon code [8, 9] proves fault tolerance by solving a system of linear equations. In this case, the matrix inverse method is used to solve for complete symbols (full strips in our terminology) under maximum failures. In contrast, our method addresses individual bits in symbols (i.e., elements) for any distribution of erased bits (within or beyond symbol fault tolerance). The binary BR [3] codes have a recursive solution to two full strip losses; the authors provide a closed form solution to the recursion. For the EVENODD code [2], the authors give a recursion and point out that it could be solved explicitly. An explicit solution to the recursion is equivalent to our matrix solution in the special case of full strip losses (again, our method has no such correlation requirements). The BCP [1], ZZS [11], X-code [10], and RDP [4] codes all have recursive reconstruction algorithms. The latter two (as well as the EVENODD code) are "geometric" and easy to visualize; the former are more "combinatorial" and less intuitive. In either case, these codes with recursive reconstruction algorithms are well-suited to our hybrid methodology. In addition, a variant of our hybrid method applies to any erasure codes suitable for disk arrays, with or without a recursive reconstruction algorithm.

## 3 Binary Linear Algebra – A Review

In this section we recall and elaborate on some basic notions from the theory of linear algebra over a binary field (which is assumed for all operations from now on without further comment – the theory extends easily to non-binary fields as well). A set of binary vectors is linearly independent if no subset sums modulo 2 to the zero vector. Let $G$ be a rectangular matrix of size $N \times M$ with $N \leq M$. The "row rank" of $G$ is the maximum number of linearly independent row vectors. The matrix $G$ has "full row rank" if the row rank equals $N$ (the number of rows). A "null space" for $G$ is the set of *all* vectors that are orthogonal (have zero dot-product) with *every* row vector of $G$. This is a vector space closed under vector addition modulo 2. A "null space basis" is a maximal set of linearly independent vectors from the null space. If the null space basis has $Q$ vectors, then the entire null space has $2^Q - 1$ total non-zero vectors.

We will write the null space vectors as column vectors, to make matrix multiplication simpler to write down, though this is not the standard convention.

Let $B$ be a basis for the null space of $G$. More precisely, $B$ is a matrix whose columns form a basis for the null space. If $G$ has full row rank, then $B$ has dimensions $M \times Q$ where $Q = M - N$.

Suppose $G$ is full row rank. A "right pseudo-inverse" is a matrix $R$ (of size $M \times N$) so that

$$G \cdot R = I_N$$

where $I_N$ is the $N \times N$ identity matrix. If $M = N$, then $R$ is the *unique* inverse. A right pseudo-inverse must exist if $G$ has full rank and is never unique if $N < M$.

More generally, let $G$ have row rank $K \leq N$, then a "partial right pseudo-inverse" (or partial pseudo-inverse) is a matrix $R$ so that

$$G \cdot R = J_K$$

where $J_K$ is an $N$-dimensional square matrix with $K$ ones on the diagonal, $N - K$ zeros on the diagonal and zeros elsewhere. Note that $R$ is a partial pseudo-inverse if the product $G \cdot R$ has a maximal number of ones over all possible choices for $R$. If $G$ is full row rank then $K = N$, $J_N = I_N$ and $R$ is a (complete) pseudo-inverse. The matrix $J_K$ is unique; that is, the positions of zero and non-zero diagonal elements are determined from $G$ and are independent of the choice of $R$.

Let $B$ be a $M \times Q$ basis for the null space basis for $G$ (perhaps padded with all-zero columns), and $R$ some *specific* partial pseudo-inverse for $G$. As $X$ varies over all binary $Q \times N$ matrices, we have

$$G \cdot (R + (B \cdot X)) = J_K. \tag{1}$$

so $R + (B \cdot X)$ runs over all partial pseudo-inverses (the proof of this is a simple calculation). What $X$ does in (1) is add a null space vector to each of the columns of $R$. For our purposes, an optimal $R$ would have minimum weight (fewer ones) in each column (that is, be the most sparse). In Section 6 we discuss algorithms for computing pseudo-inverses and in Section 6.2 we discuss algorithms for finding an optimal pseudo-inverse.

For each column of $J_K$ with a zero on the diagonal, the corresponding column of $R$ can be replaced with the all-zero column without affecting the partial pseudo-inverse property and in fact such an action clearly improves the weight of $R$. Consequently, we add this property to the definition of a partial pseudo-inverse.

Strictly speaking, the term "pseudo-inverse" applies only to real or complex matrices and implies uniqueness (optimality in a metric sense). We overload the term here with a slightly different meaning – we allow for non-uniqueness and do not require optimality (most sparse).

In the next section we apply these notions to the problem of reconstruction of scattered sectors in a stripe.

## 4 Generator and Parity Check Matrices

In this section we recall the erasure code notions of "generator matrix" and "parity check matrix". These are the basic structures upon which we develop our methodology. For a basic reference, see [7].

The generator matrix $G$ of an erasure code converts the input "word" (incoming data) into a "code word" (data and parity). The parity check matrix verifies that the "code word" contains consistent data and parity (parity scrub). In the context of erasure codes for disk arrays, the generator matrix actually provides much more.

The generator matrix is given a column block structure: each block corresponds to a strip and each column within a block corresponds to an element within the strip. If the column contains only a single 1, then the element contains user data. We call such a column an "identity column" because it is a column of an identity matrix. If the column contains multiple 1s, then it corresponds to an element which is the XOR sum of some set of user data elements; that is, the element is a parity element. In other words, the generator matrix specifies the data and parity layout on the strips, the logical ordering of the strips within the stripe, *and* the equations used to compute parity values. For example, the generator matrix for the EVENODD(3,5) code with prime $p = 3$ on 5 disks is

$$G = \left(\begin{array}{cc|cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right).$$

(more details on this example are given in Section 7).

Though it is not a requirement, the generator matrix for disk arrays typically has an identity column for each user data element (so that this data is always copied to the element's sectors verbatim in some strip and can then be read with minimal IO costs). In coding theory, a generator matrix of this form is called "systematic".

Let $D$ be a row vector of input user data values, then the row vector $S$, given by the expression

$$S = D \cdot G, \tag{2}$$

represents the data and parity elements that are stored in the stripe on the disks. The vector $D$ is indexed by the logical addresses of the user data values (say, as viewed by the host). The vector $S$ represents the physical addresses of the data and parity elements, both the disk (actually, strip, identified by the block of the generator matrix) and the sector addresses on the disk (element or offset within the strip, identified by the column within the block). $S$ is also block-structured with blocks matching those of $G$. (See our example in Section 7.)

If there are $N$ data elements input into the code and $Q$ parity elements computed by the code, then the generator matrix has dimensions $N \times (N + Q)$. (Note that $N$ is the total number of data elements within a stripe, not the number of strips; similarly, $Q$ is the number of parity elements in the stripe, not the number of parity strips.)

The "parity check matrix" $H$ has dimensions $(N + Q) \times Q$ and can be derived directly from the generator matrix (and vice-versa). Communication channels use the parity check matrix to detect errors. Each column corresponds to a parity element. After the data and parity is read off the channel, the parity is XORed with the data as indicated by its corresponding column to produce a "syndrome". If a syndrome is not zero, an error has occurred (either in the received parity symbol or in one of the dependent data symbols). For erasure codes in disk arrays, this is a parity consistency check (or parity scrub). In other words, with $S = D \cdot G$ as above, the test

$$S \cdot H == 0 \tag{3}$$

is a parity consistency check.

The parity check matrix is row blocked exactly corresponding to the column blocks of $G$ (or $S$) and it can be arranged to contain an embedded identity matrix (corresponding to the parity elements) – this is easy if $G$ is systematic. The parity check matrix for the example generator matrix $G$ above is

$$H = \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

In short, the generator matrix is used to compute the data and parity (and its layout) for storage on the disks. The parity check matrix can be used when all the data and parity are read off the disk (e.g., during parity scrub) to look for errors.

If a code can tolerate $t \geq 1$ lost disks or strips, then $G$ must have the property that if any $t$ *blocks* of $G$ are removed (or zeroed), then the resulting matrix must have full row rank. The parity check matrix is full column rank (because of the embedded identity matrix).

Also, (3) implies that

$$D \cdot G \cdot H = 0$$

should hold for every data vector $D$. This means that $G \cdot H = 0$ identically, so that each vector in $H$ is in the null space of $G$. A simple dimensionality argument shows that in fact $H$ is a basis of the null space of $G$.

In addition, it should be clear that if $G$ is systematic, then there exists an $M \times N$ matrix $R_0$ containing an embedded identity matrix of size $N \times N$ so that $R_0$ is a pseudo-inverse for $G$. $R_0$ just picks off the embedded systematic portion of $G$. If $G$ is not systematic, a pseudo-inverse $R_0$ can still be constructed, but it will not be so simple (see Section 6.3).

## 5 Simulating Scattered Sector Loss and Reconstruction

In this section, we develop our theory for solving the first of our two problems: how to deal with uncorrelated sector loss. An example is given in Section 7

We indicated above that a $t$-fault-tolerant code $G$ must have the property that zeroing any $t$ *blocks* of $G$ should leave $G$ full rank so that a complete pseudo-inverse for $G$ must exist. This suggests that we can simulate correlated *and/or* uncorrelated sector loss by zeroing or removing the associated *individual* columns from $G$. It should be clear that certain combinations of uncorrelated sector losses will result in some or all data loss events (some or all lost sectors having unrecoverable data); other combinations may involve no data loss events. Our methodology will determine, in a straightforward manner, exactly what sectors become data loss events and for those that do not, will provide a reconstruction formula for the data from these sectors.

Suppose we detect a set $F$ of failed sectors in a stripe (correlated, perhaps because of disk failure, or uncorrelated, because of medium errors, or a combination of these). Completely ignoring the block structure of $G$, let $\widehat{G}$ be a version of a generator matrix $G$, with zeroed columns corresponding to the sectors in $F$. Suppose we can find a matrix $R$ of size $M \times N$ so that

- $R$ is a partial pseudo-inverse of $\widehat{G}$, and
- $R$ has zeros in all rows that correspond to the lost columns of $\widehat{G}$.

We associate the columns of $R$ to the user data values in $D$. In Section 6 we discuss algorithms for constructing $R$. The following theorem contains our main theoretical result:

**Theorem 1.** *Let $G$, $\widehat{G}$, and $R$ be as above. Any theoretically recoverable user data value corresponds to a non-zero column of $R$ and the non-zero bit positions indicate the data and parity elements whose XOR sum equals the data value. As a special case, a directly readable data value corresponds to an identity column in $R$. A data loss event (unrecoverable data value) corresponds to an all-zero column of $R$.*

**Proof.** Let $\widehat{S}$ be the vector $S$ as in (2) but with zeros in the positions corresponding to the lost elements (the zeroed columns of $G$). Then it is clear that

$$D \cdot \widehat{G} = \widehat{S}.$$

Consequently, we have

$$\widehat{S} \cdot R = D \cdot \widehat{G} \cdot R = D \cdot J_K = \widehat{D},$$

where $\widehat{D}$ is the vector $D$ with zeros in all locations corresponding to zero's on the diagonal of $J_K$ which also corresponds to the all-zero columns of $R$.

The fact that $J_K$ is uniquely determined by $\widehat{G}$ means that any zero diagonal entry of $J_K$ induces a zero in $\widehat{D}$; this corresponds to a data loss event. Any non-zero diagonal entry of $J_K$ induces a non-zero (not identically zero) data value in $\widehat{D}$. But the non-zero diagonal entries of $J_K$ corresponds to non-zero columns of $R$ and the zero diagonal entries correspond to all-zero columns of $R$. This proves part of the first and last statements.

Now consider a non-zero column of $R$. Each non-zero bit in such a column selects into an XOR formula a data or parity element from $\widehat{S}$. Because $R$ has zeros in row positions corresponding to zeroed positions in $\widehat{S}$, such a formula does not depend on any lost data or parity element. The XOR formula then indicates that a specific XOR sum of known data and parity elements equals the data value associated to that column. That is, such a column provides a formula for the reconstruction. This proves the rest of the first statement in the theorem. The second claim of the theorem is clear. □

We emphasize that this theorem makes no assumptions about the location of the failed sectors, whether they are correlated, uncorrelated or some of both. Consequently, the theorem can be applied to the case of full disk/strip losses (highly correlated) or even to the case where there is a lost sector on *every* strip (highly uncorrelated). It also does not depend on any special structure (for example geometric layout) of the erasure code. All the information we need is embedded within the generator matrix.

Recall that $R$ is not necessarily unique and that given a basis for the null space of $\widehat{G}$, it is easy to construct, as in (1), other pseudo-inverses that satisfy the same properties as $R$ in the theorem. In the next section, we discuss methods for constructing pseudo-inverses and bases for null spaces. We use the null space bases for improving the sparseness of the pseudo-inverse.

## 6 Pseudo-inverse Constructions

There are many possible algorithms for computing pseudo-inverses and null space bases. Fundamentally, they are equivalent though the data structures and approaches differ somewhat.

From now on, we use the label $B$ to indicate a matrix whose columns form a null space basis for some zeroed matrix $\widehat{G}$, perhaps with all-zero column vectors as padding. Furthermore, because we are concerned only with uncorrelated sector loss, we ignore the block structure of $G$. As a result, we can assume without loss of generality that the generator matrix $G$ has its systematic identity submatrix in the first $N$ columns, with the parity columns in the right most $Q$ columns – we call this "left systematic". (If not, a permutation of the columns of $G$ and corresponding column positions in $\widehat{G}$, $S$, and $\widehat{S}$ and row positions of $B$, $H$ and $R$ will reduce us to this case.)

The input to our algorithms is the original generator matrix $G$ (and/or its parity check matrix $H$) and a list $F$ of data or parity elements which are declared lost (unreadable) in the stripe.

The output of our algorithms will be two matrices $R$ and $B$: $R$ is a pseudo-inverse of $\widehat{G}$ (obtained from $G$ by zeroing the columns of $G$ corresponding to the elements in $F$) and $B$ is a basis for the null space of $\widehat{G}$.

Our algorithms use "column operations" and/or "row operations" to manipulate matrices. Columns operations are equivalent to right multiplication by simple matrices (for rows, the operations are on the left). We consider three simplified column (or row) operations:

- Swap: exchange two columns (or rows)
- Sum and Replace: add column $c$ to column $d$ (modulo 2) and replace column $d$ with the sum (similarly for rows).
- Zero: zero all the entries in a column (or row).

The first two are invertible (reversible), the Zero operation is not.

Our preferred algorithm, called the "Column-Incremental" construction, can be viewed as a dynamic or on-line algorithm. It progressively updates data structures as new lost sectors are detected (simulated by a incremental processing of the elements in $F$). In Section 6.3, we outline some additional constructions including static or off-line algorithms.

## 6.1 Column-Incremental Construction

The algorithm presented here is an incremental algorithm. It starts with a pseudo-inverse and null space basis for the matrix $G$ (in the "good" state) and incrementally removes (simulates) a lost data or parity element, while maintaining the pseudo-inverse and null space basis properties at each step. The algorithm is space efficient and for most well-designed codes, has relatively few operations. It requires space in $R$ only for the lost data elements (there is no need to provide recovery formulas for parity elements as these can be easily derived from the original formulas in the generator matrix – alternatively, parity columns may be added to $R$ and so provide additional formulas for a parity computation that reflect the lost data elements). For clarity of exposition, our description is not optimally space efficient; we leave that to the expert implementor.

The process is reversible so long as the pseudo-inverse has full rank; that is, at any step, it is possible to model reconstruction of data values for lost elements (in any order) and compute a new pseudo-inverse and null space basis equivalent to one in which the recovered elements were never lost. This is described in Section 6.4

In this algorithm, column operations are performed on a workspace matrix. The lost data or parity elements index a row of $R$ and $B$.

**Algorithm: Column-Incremental Construction**

1. Construct a square workspace matrix $W$ of size $(N + Q)$. In the first $N$ columns and rows, place an identity matrix. In the last $Q$ columns, place the parity check matrix $H$. Let $R$ represent the first $N$ columns and $B$ represent the last $Q$ columns of $W$, so $W = (R \mid B)$, where initially, $B = H$ and $R = \begin{pmatrix} I_N \\ 0 \end{pmatrix}$.
2. For each lost element in the list $F$, let $r$ indicate the row corresponding to the lost element; perform the following operation:
   (a) Find any column $b$ in $B$ that has a one in row $r$. If none exists, Zero any column in $R$ that has a one in row $r$ and continue to the next lost element. (Note that zeroing these columns zeros the entire row $r$ in $W$.)
   (b) Sum and Replace column $b$ into every column $c$ of $W$ (both $R$ and $B$ portions) that has a one in row $r$.
   (c) Zero column $b$ in $B$; equivalently, add column $b$ to itself. Continue to the next lost element, until the list $F$ has been processed.
3. (Optional) Use the columns of $B$ to improve the weight of non-trivial columns of $R$ (corresponding to lost data elements processed so far). See equation (1) and Section 6.2.
4. **Output** $R$ (the first $N$ columns of $W$) and the non-zero columns of $B$ (from the last $Q$ columns of $W$).

A proof that this algorithm satisfies the required properties can be found in the appendix of the full technical report [5]. We make the following observations.

• In practice, the workspace matrices are not very large. For example, the EVENODD code on 8 strips (with prime $p = 7$) and 16 strips (with $p = 17$) consumes only 288B and 8KB, respectively. In addition, the operations are XOR or simple pointer operations, so implementation can be very efficient. On the other hand, the invocation of this algorithm happens in an error code-path, so performance is less important than

meeting the User Contract set forth in Section 1.

• The runtime complexity of the algorithm (excluding the optimizing step 3) can be bounded by $O(|F| \cdot M^2)$ bit operations since at each of the $|F|$ steps, at most $M$ ones can appear in row $r$ and each such one induces $M$ bit operations (one column operation). This is clearly an excessive upper bound as generally the matrices will be very sparse and only very few (typically $O(t)$ or $O(Q)$) ones will be in each row.

• The optimizing step 3 on $R$ can be done either as given in a penultimate or post-processing step or during the loop after step 2c. Preferably, it is done post-processing as this step can be quite expensive (see Section 6.2). It can also be skipped; it is used to possibly minimize the XOR/IO costs but is not necessary to meet the requirements of the User Contract.

• At step 2a, there may (most likely will) be multiple choices for the column. There is no known theory that provides a criterion so that the resulting $R$ is optimal or near optimal. One heuristic (the greedy-choice) is to use the column in $B$ of minimal weight, but that has not always precluded a post-processing step 3 in our experiments. However, this approach does introduce the optimal formula for the current lost element (though this may change at later rounds of the loop).

An alternative heuristic is the following: in the algorithm, a column $b$ of $B$ is chosen with a one in position $r$ among all such columns of $B$. This selected column is added to each of the others in $B$. This suggests that a heuristic for $b$ is to pick the one that minimizes the total weight of the resulting columns. In 2-fault-tolerant codes, there are typically at most two such columns to choose from, so this approach is equivalent to the one of minimal weight above; this is not true for higher fault-tolerant codes.

• For only data elements (and systematic codes), it is always the case that column $c = r$ has a 1 in position $r$ (and no other 1s elsewhere) so is always acted on in the key step. In fact, the result for this column is that we replace this column by the parity column $b$ and then toggle the bit off in position $r$.

• We claim that after each lost element in the list is processed, the matrix $R$ is a (partial or complete) pseudo-inverse for a zeroed generator matrix $\widehat{G}$ that has exactly the columns zeroed corresponding to the set of elements processed so far. This is clear in the first step because no elements have been processed, $\widehat{G} = G$, the generator matrix, $R$ is essentially an identity matrix which extracts the identity portion of $G$ and $B = H$ is the parity check matrix, *a.k.a.* the null space basis for $G$. The fact that this holds true at every other step will become clear from the proof (see the appendix in the full technical report [5]).

• We never actually write down the intermediate (or final) matrix $\widehat{G}$. This is all handled implicitly, and so no space is used for this purpose.

• Because we perform only column operations on $R$, it is easy to see that what we are doing is performing, in parallel, the operations needed to determine a reconstruction formula for all lost data elements. That means that one could perform this process on individual columns as needed (e.g., to recover a single element on-demand). This would be fairly expensive globally because one repeats the same search and process algorithm on $H$ each time, but may be marginally quicker if only one column is really needed.

• For the same reason, given the list of lost elements $F$, one can operate only on these columns in $R$ and ignore all other columns. In our construction, we use all columns because in principle, we do not know what column is coming next (the algorithm does not care), so we operate on all of $R$ at once.

• The algorithm can be used in an on-line fashion to maintain recovery formulas for lost data elements as they are detected in the stripe. As each new loss is detected, the matrices $R$ and $B$ get updated. If a lost element's value is reconstructed, the algorithm of Section 6.4 may be applied to again update these matrices to incorporate this new information. Alternatively, the algorithm can be applied as an off-line algorithm and applied after detection of all lost elements in the stripe.

This algorithm was a key ingredient to the results of [6] where it was applied to measure performance costs for a large variety of very different 2-fault-tolerant codes.

## 6.2 Improving a Pseudo-inverse

In this section we outline some approaches to implementing the optimizing step 3 in the Column-Incremental construction algorithm given above. As noted earlier, this step is not required to meet the User Contract stated in Section 1.

The following algorithm provides a systematic (though potentially very expensive) approach to finding an optimal $R$.

**Algorithm: Improve** $R$

1. Compute *all* the null space vectors (by taking all possible sums of subsets of the basis vectors).
2. For each non-identity (and non-zero) column of $R$, do the following:
   (a) For each null space vector (from step 1), do the following:
      i. Add the null space vector to the column of $R$ to generate a new formula.
      ii. If the formula generated has lower weight, then replace it in $R$.
3. **End**

Of course, this is only practical if the null space has small enough basis set. If the null space basis has very few vectors, then this algorithm provides an exhaustive search solution to finding an optimal $R$. In general, one can use any subset of the full null space to find better, but perhaps not optimal, pseudo-inverses (in Step 1 above, compute only some subset of the null space). One simple choice, is to use only the basis vectors themselves, or perhaps the basis vectors and all pairwise sums. It is an open mathematical question if there are better algorithms for finding the optimal $R$ than that given here. However, for the extensive experiments we ran for [6], the difference between optimal and near optimal was quite minimal.

## 6.3 Alternative Constructions

There are alternative constructions that can be applied to computing pseudo-inverses. Among them is a Row-Incremental variation that is analogous to the Column-Incremental method described above but uses row operations instead of column operations. Most of the steps are the same as for the Column-Incremental construction. At step 2b, for each one in positions $s \neq r$ in the selected column $b$ of $B$, Sum and Replace row $r$ into row $s$ of $B$; mirror this operation in $R$. At step 2c zero row $r$ in $B$ and $R$ and proceed to the next lost element. This algorithm has all the same properties as the column variation (including reversibility), but is typically more expensive, requiring more row operations.

Alternatively, there are both column and row versions that parallel the classical algorithm for computing an inverse. Namely, start with two matrices, the original generator matrix and an $(N + Q)$-identity matrix. Zero the columns of the generator matrix and the identity matrix corresponding to each lost data and parity element. Perform column (or row) operations on the modified generator matrix to convert it to column (or row) reduced echelon form. Parallel each of these operations on the identity matrix; the resulting matrix contains both the pseudo-inverse and null space basis. These variations are static, off-line constructions as they utilize the complete set of lost elements in the very first step. As before, the column version has marginally less computation.

We do not give proofs for any of these constructions as they vary only slightly from the proof of the Column-Incremental construction found in the appendix of the full technical report [5]. The static algorithms can also be used to construct an initial pseudo-inverse matrix for the full generator matrix in the case when $G$ is not systematic.

## 6.4 Reversing The Column Incremental Construction

As mentioned, the incremental process can be used to start with a fully on-line stripe and, step by step, as medium errors are detected in the stripe, maintain a set of reconstruction formulas (or a declaration of non-reconstructability) for every data element in the stripe. As new medium errors are detected, the matrices are updated and new formulas are generated.

It might be useful to reverse the process. Suppose the array has had some set of medium errors, but no data loss events and suppose a data element is reconstructed by its formula in $R$. If this reconstructed data is replaced in the stripe, it would be helpful to update the formulas to reflect this. There are two reasons for this. First, we know we can replace the formula in $R$ by an identity column (we no longer need the old formula). But second, it may be the case that other lost elements can be reconstructed by better formulas that contain this newly reconstructed element; we should update $R$ to reflect this fact.

One approach would be to use any algorithm to recompute from scratch the formulas for the revised set of sector losses. However, the incremental algorithm suggests that we might be able to reverse the process; that is, to update $R$ and $B$ directly to reflect the fact that the data element has been reconstructed (e.g., its column in $R$ is replaced by an identity column).

To fully reverse the incremental construction of the previous section, it must be the case that no information (in the information-theoretic sense) is lost through each step. Mathematically, this happens whenever we perform a non-invertible matrix operation, i.e., that corresponds to multiplication by a non-invertible matrix. This occurs essentially in only one place in the construction: whenever we can find no vector in the null space basis with a one in the desired row. This corresponds exactly to the case where we have data loss events.

Consequently, we have the following result: so long as we never encounter the data loss branch, then (in principle), the sequence of steps can be reversed. However, the algorithm we give below works even after data loss events, so long as the restored element has a reconstruction formula in $R$, i.e., it is *not* itself a data loss event . Note that it makes little sense to consider restoring into the matrix an element corresponding to a data loss event (the theorem says that this is theoretically impossible).

The algorithm below performs this incremental restoration step in the case of a (recoverable) data element. Section 6.4.1 discusses the parity element case.

The input to this algorithm is a workspace matrix $W = (R \mid B)$ (possibly) generated by the incremental algorithm and having the property that

$$\widehat{G} \cdot W = (I_N \mid 0)$$

where $\widehat{G}$ is the generator matrix with zeroed columns for each data or parity element in the set $F$ of assumed lost elements (prior to a reconstruction). The other input is a data element index, that is, a row number $r \leq N$ of $W$. The output of the algorithm is a revised matrix $W$ so that the above formula holds with $\widehat{G}$ having column $r$ replaced by the appropriate identity column. The new matrix $W$ will have an identity column in position $r$. (As before, the algorithm does not track the changes to $\widehat{G}$ directly, only implicitly.) Note that this process does not depend on which element is being restored from among the set of elements removed during the incremental phase (that is, it need not be the last element removed). We assume that $B$ contains enough all-zero columns so that it has $Q$ columns in total.

If the restored element is not from the set $F$, then this algorithm has no work to do, so we assume that the lost element is from $F$.

**Algorithm: Reverse Incremental Construction**

1. (Optional) For each column $c$ in the inverse portion of $W$ (first $N$ columns) that has a one in every row that column $r$ has (that is, if the AND of the columns $c$ and $r$ equals column $r$), do the following:
   (a) Sum and Replace column $r$ into column $c$; that is, for each position of column $r$ that has a one, set the corresponding value in column $c$ to zero.
   (b) Set position $r$ in column $c$ to the value 1.
2. Find any all-zero column $b$ in the null space portion of $W$ (in the last $Q$ columns).
3. Set position $(r, r)$ and $(r, b)$ in $W$ to the value 1.
4. Swap columns $r$ and $b$ of $W$.
5. (Optional) Use the null space basis vectors in $B$ of $W$ to reduce the weight of any column in the inverse portion $R$ of $W$.
6. **Return** the updated $W$.

This algorithm works because it takes the reconstruction formula for the data element and unfolds it back into the null space basis, then replaces the formula with an identity column.

The first optional step replaces any occurrence of the formula for data element $r$ in the original $W$ by that element itself. In particular, it explicitly restores into other columns a dependence on the restored data element. In the process, it improves the weight of these formulas.

This algorithm does not necessarily completely reverse the incremental algorithm in that it does not necessarily produce identical matrices going backward as were seen going forward. However, the difference will always be something in the null space.

A proof of this construction is given in the appendix of the full technical report [5].

### 6.4.1 Restoring parity elements

To add a parity element back in to the matrices, we need to have the original parity column from the generator matrix $G$ (for the data columns, we know *a priori* that this column is an identity column so we do not need to keep track of this externally). Suppose that this parity is indexed by column $c$ in $G$.

Take this parity column and for each 1 in the column, sum together (modulo 2) the corresponding columns of $R$ in $W$ and place the result in an all-zero column of $B$ in $W$. (This is exactly what we did for a data column since there was only one such column!) Replace the zero in position $c$ of this new column by 1. Replace column $c$ of $G_0$ by this parity column (restore it). (Again, this is exactly what we did for a restored data column, except we also had to set the $(r, r)$ position in the inverse portion of $W$ to 1 – in the case of a parity column, no such position exists in the inverse portion so this step is skipped.)

A proof is given in the appendix of the full technical report [5].

## 7 An Example: EVENODD Code

Consider the EVENODD(3,5) code [2] with prime $p = 3$, $n = 5$ total disks, $n - 2 = 3$ data disks and two parity disks. The data and parity layout in the strips and stripe for one instance is given in the following diagram:

| $S_0$ | $S_1$ | $S_2$ | $P_0$ | $P_1$ |
|---|---|---|---|---|
| $d_{0,0}$ | $d_{0,1}$ | $d_{0,2}$ | $P_{0,0}$ | $P_{0,1}$ |
| $d_{1,0}$ | $d_{1,1}$ | $d_{1,2}$ | $P_{1,0}$ | $P_{1,1}$ |

The columns labeled $S_0, S_1, S_2$ are the data strips in the stripe (one per disk); the columns labeled $P_0$ and $P_1$ are the horizontal and diagonal parity strips, respectively. We order the data elements first by strip and then, within the strip, down the columns (this is the same view as the ordering of host logical blocks within the stripe). In this example, $N = 6$ and $Q = 4$.

The generator matrix $G$ defined for this code is:

$$G = \left(\begin{array}{cc|cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right).$$

This is column blocked to indicate the strip boundaries. The matrix indicates that the parity $P_{0,1}$ is the XOR sum of the data elements indexed by the 0th, 3th, 4th and 5th rows of $G$, i.e.,

$$P_{0,1} = d_{0,0} + d_{1,1} + d_{0,2} + d_{1,2}. \quad (4)$$

The parity check matrix $H$ is:

$$H = \left(\begin{array}{cccc} 1&0&1&0\\ 0&1&0&1\\ \hline 1&0&0&1\\ 0&1&1&1\\ \hline 1&0&1&1\\ 0&1&1&0\\ \hline 1&0&0&0\\ 0&1&0&0\\ \hline 0&0&1&0\\ 0&0&0&1 \end{array}\right).$$

The parity check matrix is row blocked exactly to correspond to the column blocks of $G$ and it contains in the lower portion an embedded identity matrix. It is easy to see that $G \cdot H = 0$; that is, $H$ is in the null space of $G$ (and forms a basis as well). Each column of the parity check matrix corresponds to a parity value in the array (the identity rows and the block structure provide this association).

For example, column 3 of the parity check matrix says

$$d_{0,0} + d_{1,1} + d_{0,2} + d_{1,2} + P_{0,1} = 0.$$

If this equation is not satisfied for the actual data and parity read from the disks (or detected on a channel), then an error has occurred somewhere.

More generally, we interpret these matrices in the following way. As labeled above, we consider the user data values as a row vector (ordered as already indicated):

$$D = (d_{0,0}, d_{1,0} | d_{0,1}, d_{1,1} | d_{0,2}, d_{1,2}).$$

The product $S = D \cdot G$ equals

$$(d_{0,0}, d_{1,0} | d_{0,1}, d_{1,1} | d_{0,2}, d_{1,2} | P_{0,0}, P_{1,0} | P_{0,1}, P_{1,1})$$

indicates the data layout in strips (via the block structure) as well as the formulas for computing the parity. We saw an example of this in equation (4).

The parity check matrix implies that

$$S \cdot H = 0,$$

regardless of the actual values of the data elements.

Any binary linear combination of the columns of $H$ will also be orthogonal to all the vectors in $G$. E.g., take the binary sum (XOR) of columns 0 and 3 in $H$:

$$(1, 1 | 0, 1 | 0, 0 | 1, 0 | 0, 1)^t.$$

It is easy to see that this has the desired orthogonality property. We can replace any column in $H$ by any such combination and still have a "parity check matrix". Typically, the $H$ constructed directly from the parity equations is the most sparse.

## 7.1 The Example – Scattered Sector Loss

Suppose we loose strip $S_0$ and only data element $d_{0,2}$ of $S_2$ in the EVENODD(3,5) code above. We then have a "zeroed" matrix $\widehat{G}$ in the form:

$$\widehat{G} = \left(\begin{array}{cc|cc|cc|cc|cc} \bullet & \bullet & & & \bullet & & & & & \\ 0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&1&0&0&0&1&0&0&1\\ 0&0&0&1&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&1&0&1&1&0 \end{array}\right)$$

where the $\bullet$ over the column indicates the column has been removed by zeroing.

Using the data vector $D$, we see that we have a revised set of relationships:

$$D \cdot \widehat{G} = \widehat{S}, \tag{5}$$

where

$$\widehat{S} = (0, 0 | d_{0,1}, d_{1,1} | 0, d_{1,2} | P_{0,0}, P_{1,0} | P_{0,1}, P_{1,1}).$$

When we view the vector $\widehat{S}$ as "known" data and parity elements (in fact, the labeled components represent the sectors that are still readable in the stripe), this equation represents a system of linear equations for the "unknown" vector $D$ in terms of the known vector $\widehat{S}$.

The following two matrices $R$ and $R'$ are easily seen to be pseudo-inverses for $\widehat{G}$:

$$R = \left(\begin{array}{cccccc} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&1&0&1&0\\ 0&1&0&1&0&0\\ \hline 0&0&0&0&0&0\\ 1&1&0&0&1&1\\ \hline 1&0&0&0&0&0\\ 1&1&0&0&1&0\\ \hline 0&0&0&0&0&0\\ 1&0&0&0&1&0 \end{array}\right), \quad R' = \left(\begin{array}{cccccc} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&1&0&0&0\\ 0&1&0&1&1&0\\ \hline 0&0&0&0&0&0\\ 1&1&0&0&0&1\\ \hline 1&0&0&0&1&0\\ 1&1&0&0&1&0\\ \hline 0&0&0&0&1&0\\ 1&0&0&0&1&0 \end{array}\right). \tag{6}$$

We show how these matrices are obtained in Section 7.2.

The columns of $R$ (or $R'$) correspond to the data elements as ordered in the vector $D$. Each non-zero row corresponds to a position in the vector $\widehat{S}$ of known elements. Each all-zero row matches a lost element in $\widehat{S}$. Each column represents an XOR formula for reconstructing the data element to which it corresponds. For example, to reconstruct $d_{0,2}$, we look at column 4 of $R$. It indicates the following formula:

$$d_{0,2} = d_{0,1} + d_{1,2} + P_{1,0} + P_{1,1},$$

and by looking at column 4 of $R'$ we get the formula:

$$d_{0,2} = d_{1,1} + P_{0,0} + P_{1,0} + P_{0,1} + P_{1,1}.$$

It is easy to see from the original code that both of these formulas are correct (and that they do not depend on any lost sectors!).

Because the code is MDS and can tolerate two disk/strip failures, it is easy to see from dimension counting that $\widehat{G}$ has only one non-zero vector in its null space. This vector turns out to be

$$(0,0|1,1|0,1|1,0|1,0)^t. \tag{7}$$

This is also the sum of columns 4 of $R$ and $R'$ (indicating that $R'$ is derived from $R$ by adding a vector from the null space).

The weight of each of the formulas for reconstructing data via $R$ is at least as good as those in $R'$, consequently, $R$ is a better solution than $R'$ for our purposes. In fact, with only one vector in the null space, it is clear that $R$ is optimal.

## 7.2 The Example – Constructing $R$

We start with the EVENODD(3,5) code as before and assume as above that data elements $d_{0,0}$, $d_{1,0}$, and $d_{0,2}$ are lost from strips $S_0$ and $S_2$. These elements correspond to columns $r = 0, 1, 4$ of $G$ (and also to this set of rows in our workspace).

The initial workspace is

$$W = (R \mid B) = \left(\begin{array}{cccccc||cccc} 1&0&0&0&0&0&1&0&1&0\\ 0&1&0&0&0&0&0&1&0&1\\ \hline 0&0&1&0&0&0&1&0&0&1\\ 0&0&0&1&0&0&0&1&1&1\\ \hline 0&0&0&0&1&0&1&0&1&1\\ 0&0&0&0&0&1&0&1&1&0\\ \hline 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ \hline 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right).$$

For row $r = 0$, we find some column in $B$ that has a one in this row. There are two choices, $b = 6$ or $b = 8$. We choose $b = 6$ because its weight is less. We add this to columns $c = 0$ and $c = 8$ (where there is a one in row 0), then zero column $b = 6$. The result is

$$W = \left(\begin{array}{cccccc||cccc} 0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&1\\ \hline 1&0&1&0&0&0&0&0&1&1\\ 0&0&0&1&0&0&0&1&1&1\\ \hline 1&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&1&1&0\\ \hline 1&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ \hline 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right).$$

For $r = 1$, select column $b = 7$ (again, this has the minimum weight), then add this to columns $c = 1, 9$, then zero column $b = 7$. This gives:

$$W = \left(\begin{array}{cccccc||cccc} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&1&0&0&0&0&0&1&1\\ 0&1&0&1&0&0&0&0&1&0\\ \hline 0&0&0&0&1&0&0&0&0&1\\ 1&1&0&0&0&1&0&0&1&1\\ \hline 0&0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0&0&1\\ \hline 1&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right).$$

Similarly, for $r = 4$ (using $b = 9$), the result is

$$W = \left(\begin{array}{cccccc||cccc} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&1&0&1&0&0&0&1&0\\ 0&1&0&1&0&0&0&0&1&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&1&1&0&0&1&0\\ \hline 1&0&0&0&0&0&0&0&1&0\\ 1&1&0&0&1&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&0&0 \end{array}\right).$$

Note that the left portion of this workspace equals $R$ in (6). Furthermore, our null space basis $B$ contains only the vector in (7); adding this vector to column 4 of $W$ produces $R'$ from (6). As $R$ contains the optimal reconstruction formulas, no post-process step is required in this example.

It can be checked that at each stage the claimed properties of pseudo-inverse and null space of the intermediate results all hold. It should be noted that this is not against the final $\widehat{G}$ but the intermediate $\widehat{G}$ which we never write down).

## 7.3 The Example – Additional Sector Loss

Now suppose in addition that element $d_{0,1}$ of strip $S_1$ is also lost. This corresponds to a situation where sectors are lost from all three data strips of the stripe. Nominally, the EVENODD(3,5) code can only protect against losses on 2 strips; we have three partial strips, a case not covered in the literature.

The element $d_{0,1}$ corresponds to $r = 2$. We select column $b = 8$, perform the operations in the algorithm and the result is

$$W = \left(\begin{array}{cccccc||cccc} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&1&1&1&1&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&1&0&0&0&0\\ \hline 1&0&1&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&0&0&0\\ \hline 0&0&1&0&1&0&0&0&0&0\\ 1&0&0&0&1&0&0&0&0&0 \end{array}\right). \tag{8}$$

At this point, we have no more null space basis vectors ($B$ is all zero). Any further sector loss implies a "data loss event" (see below).

Observe that any column corresponding to a data element that is *not* lost has remained unchanged as an identity column. In addition, even though we have lost sectors in three strips, all sectors are still recoverable.

If we *further* assume that data element $d_{1,1}$ (corresponding to row $r = 3$) is also lost, we can continue the algorithm. In this case, there is no null space basis vector with a one in this row. So, the algorithm says to zero all columns in $R$ with a one in this row (that is, columns $1, 2, 3, 4$). This produces the matrix

$$W = \left(\begin{array}{cccccc||cccc} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0&0&0\\ \hline 1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{array}\right).$$

This indicates that data elements corresponding to columns $1, 2, 3, 4$ are "data loss events". However, column $0$ corresponding to data element $d_{0,0}$ is still recoverable (as is $d_{1,2}$ which was never lost).

## 7.4 The Example – Reversing The Construction

We start with the result of our incremental construction example in equation (8) where we have lost sectors $d_{0,0}$, $d_{1,0}$, $d_{0,2}$ and $d_{0,1}$ corresponding to columns $r = 0, 1, 4, 2$ of $G$. Suppose we have reconstructed data element $d_{0,0}$ of column $r = 0$ (which is *not* the last element we simulated as lost). The reverse incremental algorithm above has the following steps. (We include the optional steps for completeness.)

First, we examine each of the first six columns to see if column $r = 0$ is contained in it. Column $r = 0$ has one's in positions $5, 6, 7, 9$. No other column has ones in all these positions, so we continue to the next step.

Next we select the all-zero column $b = 6$ and set position $0$ in this column and in column $r = 0$ to the value $1$, then we swap these two columns:

$$W = \left(\begin{array}{cccccc||cccc} 1&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&1&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&1&1&0&0&0\\ \hline 0&0&1&0&1&0&1&0&0&0\\ 0&0&0&0&1&0&1&0&0&0\\ \hline 0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0 \end{array}\right).$$

Next we look for null space basis elements (there's only one to choose from) that might improve the inverse portion. For example, column $4$ has weight $5$. If we combine (XOR) columns $4$ and $6$, we get a new matrix

$$W = \left(\begin{array}{cccccc||cccc} 1&0&0&0&1&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&1&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&1&1&1&0&0&0\\ \hline 0&0&1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ \hline 0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0 \end{array}\right).$$

where the new column $4$ now has weight $4$. This step improved the weight of this column, as we wanted.

Note that our final result does have an identity column in position $0$ so we have restored this data element.

# 8 Efficient Reconstruction of Partial Strips

In this section we introduce the hybrid reconstruction method. It applies the reconstruction methodology based on the matrix method in another way to address the problem of partial strip reconstruction.

Suppose the array's erasure code can tolerate two strip failures. Most such erasure codes have a recursive algorithm defined for reconstructing the two lost strips. This can be quite efficient for rebuild of both lost strips in their entirety. The steps are generally quite simple and explicitly assume use of intermediate reconstructed elements. However, such a method will be very code-dependent; that is, the recursion will depend on the specific code layout and parity formulas. On the other hand, the matrix methodology above is completely generic. If applied without the Reverse Incremental construction, no intermediate results are used; consequently, the amount of XOR computation could be quite large compared to a recursive method. But the Reverse Incremental construction would directly take advantage of intermediate results and improve overall XOR computation costs. In fact, if applied appropriately (as a special case of our algorithm below), the matrix method (including the Reverse Incremental construction) would reduce to the recursive method in most cases (and be very similar in all others).

Now consider a host request to read a single block from one of the two lost strips (prior to completion of any background process to reconstruct the stripe). If the element is very deep into the recursion, a number of intermediate reconstructions (of lost elements) must take place; these intermediate results are not needed for the immediate host request and, though they can be cached,

are potentially extraneous work for the task at hand. The matrix method above, however, gives a (near) optimal formula for direct reconstruction of any single element without reconstruction of any additional elements.

We see that for single element reconstruction, the generic direct method of the matrix methodology is generally more efficient than the recursive method provided with a specific code. Conversely, for reconstruction of all lost elements the generally preferred method is the recursive method (either explicitly using the code's specific theory or implicitly using the matrix method together with the Reverse Incremental construction).

We now consider the problem of reconstructing a partial strip, say, to satisfy a host read for multiple consecutive blocks that span multiple elements in a strip. We assume that multiple strips are lost (though that is not a requirement at all). The above discussion suggests that neither the direct nor the recursive methods may be optimal to address this problem efficiently. We propose the following algorithm. The input to the algorithm is the set of lost sectors $F$, the parity check matrix (or the generator matrix) and a subset $T$ of $F$ containing sectors to reconstruct (we assume that no element in $T$ is a data loss event). The output is the data values for the elements in $T$. That is, $F$ is the complete set of lost sectors and $T$ is that partial set we need to reconstruct.

**Algorithm: Code-specific Hybrid Reconstruction**

1. Compute the pseudo-inverse $R$ and a (padded) null space basis for $B$ for the lost sectors $F$ (say, using the Column Incremental construction).
2. Do the following until all of $T$ has been reconstructed:
   (a) Find an unreconstructed element $t \in T$ whose reconstruction vector in $R$ has minimal weight; reconstruct the value for $t$.
   (b) Examine the recursion to see if any other element $t' \in T$ can be reconstructed by some fixed number of iterations of the recursion when starting that recursion at $t$. (e.g., for 2-fault-tolerant codes, this typically means at most two steps).
   (c) If such a $t'$ exists, reconstruct $t'$ following the recursion; set $t \leftarrow t'$ and return to step 2b.
   (d) If no such $t'$ exists, do:
      i. (Optional) Update $R$ and $B$ using the Reverse Incremental construction for all values reconstructed so far.
      ii. Return to step 2a.
3. **Return** the reconstructed values for the sectors in $T$.

Essentially, this algorithm uses the direct method to jump into the recursion at the first point the recursion intersects the set $T$ (thereby avoiding reconstruction of unneeded values). The optional step 2(d)i ensures that we have factored into the direct reconstruction formulas all values reconstructed to this point, thereby allowing these elements to be used in later reconstruction formulas (lowering XOR computational costs).

During step 2c, we can avoid physical reconstruction of intermediate steps in the recursion that are *not* in set $T$ (that is, not immediately required for the host) by logically collapsing the recursion equations. That is, we combine the steps of the recursions to get from $t$ to $t'$. This has two advantages. First, it avoids a computation and temporary memory store of any unneeded intermediate result. Second, the combination can eliminate the need for some data or parity values that appear multiply (an even number of times) in the set of recursive formulas. This avoids a possible disk read to access this data as well as the memory bandwidth costs to send this data into and out of the XOR engine multiple times.

Step 2b looks for efficient ways to utilize the recursion. If none exist, we reapply the direct method (updated, perhaps) to jump back into the recursion at some other point in $T$ of minimal direct costs.

Together, these steps enable efficient reconstruction of only those elements that are needed (those in $T$) and no others. There are two special cases: (a) if $T$ is a singleton, then this method will apply the direct method in the first step then exit; (b) if $T$ is the union of all the elements on all lost strips, then the algorithm will default to the application of the recursion alone. We see then that this algorithm interpolates between these two extremes to find efficient reconstruction of partial strips. (Note that $T$ need not be a partial strip, but that is the most likely application.)

More generically, we can apply the following algorithm as a means to efficiently solve the same problem, without reference to the specific recursion of the code (assuming it has one).

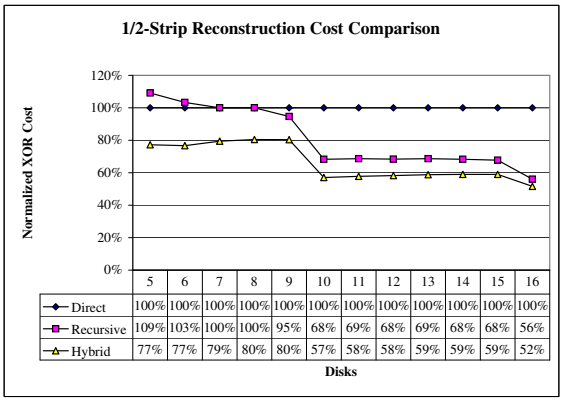
**Algorithm: Generic Hybrid Reconstruction**

1. Compute the pseudo-inverse $R$ and a (padded) null space basis matrix $B$ for the lost sectors $F$ (say, using the Column Incremental Construction).
2. Do the following until all of $T$ has been reconstructed:
   (a) Find an unreconstructed element $t \in T$ whose reconstruction vector in $R$ has minimal weight and reconstruct it.
   (b) Update $R$ and $B$ using the Reverse Incremental construction with input $t$.
   (c) Return to step 2a.
3. **Return** the reconstructed values for the sectors in $T$.

It is not hard to see that in the presense of a straight forward recursion, the code-specific and generic hybrid methods will produce similar results (perhaps in differ-

ent order of reconstruction, but with the same or similar costs). The application of the recursion in step 2c in the code-specific algorithm implicitly applies the Reverse Incremental algorithm.

Figure 1 shows the advantages of this hybrid method for the EVENODD code [2]. The chart shows the XOR costs (total number of XOR input and output variables) for disk array sizes from 5 to 16. These numbers are the average over all 1/2-strip-sized (element-aligned) host read requests to lost strips and averaged over all possible 2 strip failures. They are normalized to the Direct XOR costs. The figure shows that the direct cost is generally (except for very small arrays) more expensive than application of the recursive method (as one would expect for long reads), but it also shows that the Hybrid method is significantly more efficient than both.




| | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Direct | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |
| Recursive | 109% | 103% | 100% | 100% | 95% | 68% | 69% | 68% | 69% | 68% | 68% | 56% |
| Hybrid | 77% | 77% | 79% | 80% | 80% | 57% | 58% | 58% | 59% | 59% | 59% | 52% |

Figure 1: Comparision of Direct, Recursive and Hybrid reconstruction methods for 1/2 lost strip reconstruction, EVENODD code.

## 9 Summary

We developed a language to model uncorrelated and/or correlated loss of sectors (or elements) in arbitrary array codes. We provided a direct methodology and constructive algorithms to implement a universal and complete solution to the recoverability and non-recoverability of these lost sectors. This method and algorithm meets the User Contract that says that what is theoretically recoverable shall be recovered. Our solution can be applied statically or incrementally. We demonstrated the power of the direct method by showing how it can recover data in lost sectors when these sectors touch more strips in the stripe than the fault tolerance of the erasure code. The direct method can be joined with any code-specific recursive algorithm to address the problem of efficient reconstruction of partial strip data. Alternatively, the incremental method can be reversed when some data is recovered to provide a completely generic method to address this same partial strip recovery problem. Finally, we provided numerical results that demonstrate significant performance gains for this hybrid of direct and recursive methods.

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